Discrete nonlinear hyperbolic equations. Classification of integrable cases

V.E. Adler[1], A.I. Bobenko[2], Yu.B. Suris[3]

DDG'07, Berlin

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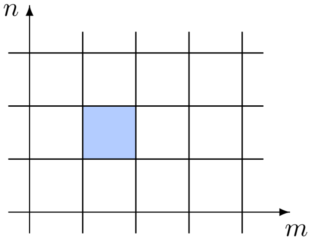
Discrete nonlinear hyperbolic equations. Classification of integrable cases



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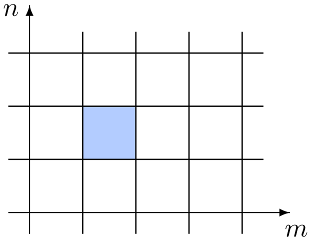

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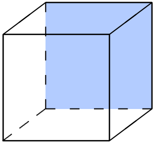
3D-consistency



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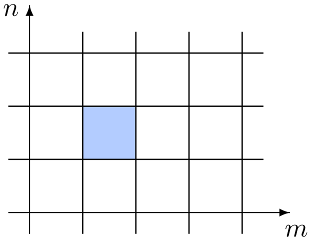

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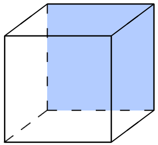

analysis of

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list of integrable equations

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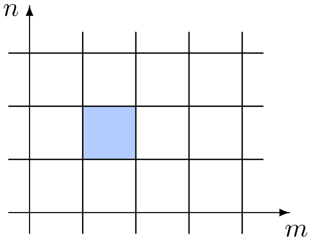

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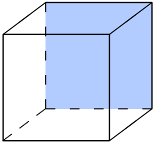

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$$Q_{m,n}(x_{m,n}, x_{m+1,n}, x_{m,n+1}, x_{m+1,n+1}) = 0, \quad x \in \mathbb{CP}^1$$

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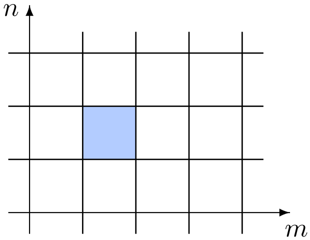

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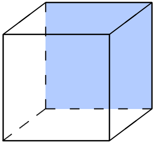

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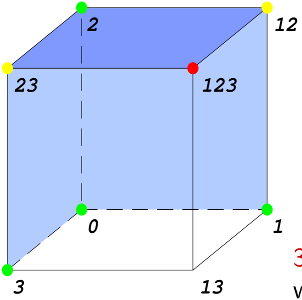

$$\begin{array}{ll} A(x,x_1,x_2,x_{12})=0 & \widetilde{A}(x_3,x_{13},x_{23},x_{123})=0 \\ B(x,x_1,x_3,x_{13})=0 & \widetilde{B}(x_2,x_{12},x_{23},x_{123})=0 \\ C(x,x_2,x_3,x_{23})=0 & \widetilde{C}(x_1,x_{12},x_{13},x_{123})=0 \end{array} \qquad (1)$$

3D-consistency: the values of x_{123} computed in 3 possible ways coincide identically on the initial values x, x_1, x_2, x_3.

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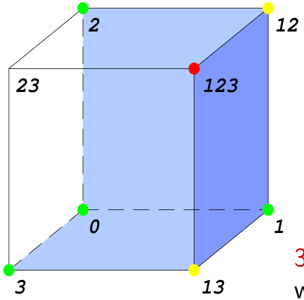

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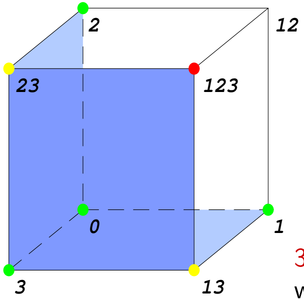

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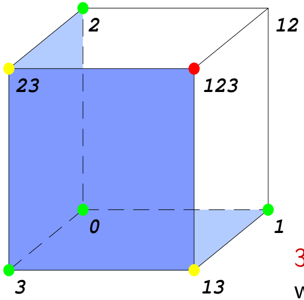

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3D-consistency: the values of x_{123} computed in 3 possible ways coincide identically on the initial values x, x_1, x_2, x_3.

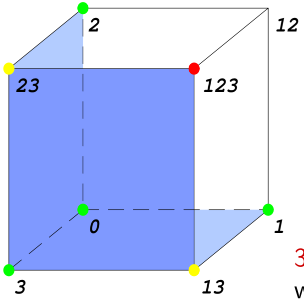
The simplest examples: discrete wave equation

$$x + x_i + x_j + x_{ij} = 0 \qquad \text{(+ the same on the opposite faces)}$$

Independently on the order of computation:

$$x_{123} = 2x + x_1 + x_2 + x_3.$$

3D-consistency



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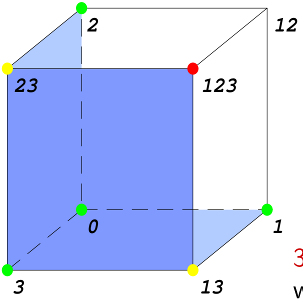
The simplest examples: Bianchi permutability theorem for sine-Gordon equation

$$\alpha^{(i)}(xx_i + x_j x_{ij}) = \alpha^{(j)}(xx_j + x_i x_{ij}) \qquad \text{(+ the same on the opposite faces)}$$

Independently on the order of computation:

$$x_{123} = -\frac{((\alpha^{(2)})^2 - (\alpha^{(1)})^2)\alpha^{(3)} x_1 x_2 + c.p.}{((\alpha^{(2)})^2 - (\alpha^{(1)})^2)\alpha^{(3)} x_3 + c.p.}$$

3D-consistency



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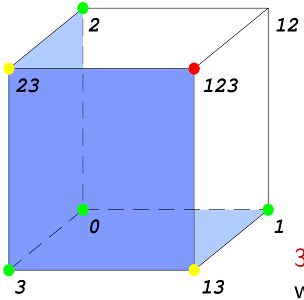
The simplest examples: discrete Korteweg-de Vries equation

$$(x - x_{ij})(x_i - x_j) = \alpha^{(j)} - \alpha^{(i)} \qquad (+ \text{ the same } \ldots)$$

Independently on the order of computation:

$$x_{123} = -\frac{(\alpha^{(2)} - \alpha^{(1)})x_1 x_2 + c.p.}{(\alpha^{(2)} - \alpha^{(1)})x_3 + c.p.}$$

3D-consistency



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The simplest examples: discrete Schwarz-KdV

$$\alpha^{(i)}(x - x_j)(x_i - x_{ij}) = \alpha^{(j)}(x - x_i)(x_j - x_{ij}) \qquad (\text{+ the same } \ldots)$$

Independently on the order of computation:

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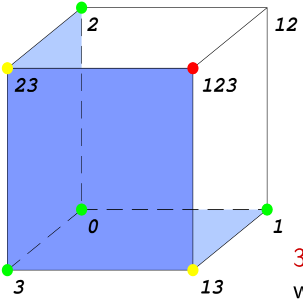
3D-consistency



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3D-consistency: the values of x_{123} computed in 3 possible ways coincide identically on the initial values x, x_1, x_2, x_3.

Our previous result was obtained under restrictive assumptions:

– equations on the opposite faces coincide (shift-invariance)

– all equations possess the square symmetry

– so-called "tetrahedron property" (x_{123} does not depend on x)

Now we deduce these properties (making some other assumption . . .)

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[4] V.E. Adler, A.I. Bobenko, Yu.B. Suris. Classification of integrable equations on quad-graphs. The consistency approach. *Comm. Math. Phys.* **233** (2003) 513–543.

Singular solutions

We consider only affine-linear equations (= of the first degree on each unknown):

$$Q(x, y, z, t) = a_1 xyzt + \cdots + a_{16} = 0. \tag{2}$$

Our approach is based on the following notion.

Definition. A solution (x, y, z, t) of equation (2) is called singular with respect to t, if it solves the equation $Q_t(x, y, z, t) = 0$ as well.

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In other words, if $Q=pt+q$ then the singular solution is characterized by the curve

$$p(x,y,z)=0, \quad q(x,y,z)=0.$$

The important role play the planar projections of this curve which are biquadratic ones:

$$h(x,y)=pq_z-p_zq=Q_zQ_t-QQ_{zt}=h_1x^2y^2+\dots+h_9=0.$$

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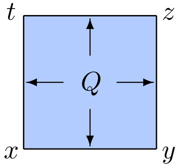
We associate such curve to each edge of the square cell which carries the equation (2):



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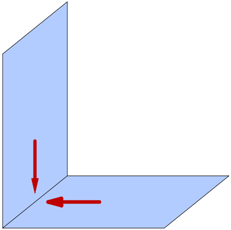
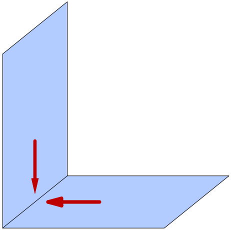


The key observation is given by the following theorem.

Theorem 1. Let the system (1) be 3D-consistent and all involved biquadratics be not degenerate. Then, for each edge of the cube, the equations corresponding to adjacent faces give rise to one and the same biquadratic curve.

The nondegeneracy assumption means that a biquadratic polynomial $h(x, y)$ must be free of the factors of the form $x - \mathrm{const}$ and $y - \mathrm{const}$ $\Longrightarrow$ two types of equations.

Singular solutions



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The idea of the proof.

Choose the singular initial data on the face $(1,2)$. This leads to an undetermined value of x_{123}. However, due to consistency, x_{123} can be found without using this face. Therefore, the initial data on the faces $(1,3)$ and $(2,3)$ must be singular as well. Therefore, the singular curves on these faces have the same projections on the common edges.

Möbius transformations

The classification is made modulo $(PSL_2(\mathbb{C}))^8$, that is the variables in all vertices of the cube are subjected to independent Möbius transformations. It is important that the following commutative diagram is compatible with the action of this group.

$$
\begin{array}{ccccc}
r_4(x_4) & \xleftarrow{\delta_3} & h^{34}(x_3,x_4) & \xrightarrow{\delta_4} & r_3(x_3) \\
\delta_1 \Big\uparrow & & \Big\uparrow \delta_{12} & & \Big\uparrow \delta_2 \\
h^{14}(x_1,x_4) & \xleftarrow{\delta_{23}} & Q(x_1,x_2,x_3,x_4) & \xrightarrow{\delta_{14}} & h^{23}(x_2,x_3) \\
\delta_4 \Big\downarrow & & \Big\downarrow \delta_{34} & & \Big\downarrow \delta_3 \\
r_1(x_1) & \xleftarrow{\delta_2} & h^{12}(x_1,x_2) & \xrightarrow{\delta_1} & r_2(x_2)
\end{array}
$$

where

$$\delta_{ij}(Q) = Q_{x_i} Q_{x_j} - Q Q_{x_i,x_j}, \qquad \delta_i(h) = h_{x_i}^2 - 2h h_{x_i,x_i}.$$

This allows to bring all r_i to several canonical forms depending on the multiplicity of zeroes:

$$(x^2-1)(k^2x^2-1), \quad x^2-1, \quad x^2, \quad x, \quad 1, \quad 0.$$

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Moreover, the examination of possible combinations is shortened if we take into account the invariants of the Möbius transformations: it can be proved that

- all polynomials r_i have the same Weierstrass invariants g_2, g_3

$$g_2 = \frac{1}{48}(2rr^{IV} - 2r'r''' + (r'')^2),$$
$$g_3 = \frac{1}{3456}(12rr''r^{IV} - 9(r')^2r^{IV} - 6r(r''')^2 + 6r'r''r''' - 2(r'')^3);$$

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- the biquadratic polynomials $h(x,y)$ also have some invariants i_2, i_3 which coincide for the polynomials on the opposite edges

$$i_2 = 2hh_{xxyy} - 2h_xh_{xyy} - 2h_yh_{xxy} + 2h_{xx}h_{yy} + h_{xy}^2,$$
$$i_3 = \det\begin{pmatrix} h & h_x & h_{xx} \\ h_y & h_{xy} & h_{xxy} \\ h_{yy} & h_{xyy} & h_{xxyy} \end{pmatrix}.$$

List of integrable equations

Theorem 2. Up to Möbius transformations, any 3D-consistent system with nondegenerate biquadratics is one of the following list ($\alpha = \alpha^{(i)}$, $\beta = \alpha^{(j)}$):

$$\operatorname{sn}(\alpha)\operatorname{sn}(\beta)\operatorname{sn}(\alpha-\beta)(k^2 x x_i x_j x_{ij} + 1) + \operatorname{sn}(\alpha)(x x_i + x_j x_{ij}) - \operatorname{sn}(\beta)(x x_j + x_i x_{ij}) - \operatorname{sn}(\alpha-\beta)(x x_{ij} + x_i x_j) = 0, \tag{Q_4}$$

$$\Big(\alpha - \frac{1}{\beta}\Big)(x x_i + x_j x_{ij}) - \Big(\beta - \frac{1}{\alpha}\Big)(x x_j + x_i x_{ij}) - \Big(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\Big)(x x_{ij} + x_i x_j) - \frac{\delta}{4}\Big(\alpha - \frac{1}{\alpha}\Big)\Big(\beta - \frac{1}{\beta}\Big)\Big(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\Big) = 0, \tag{Q_3}$$

$$\alpha(x - x_j)(x_i - x_{ij}) - \beta(x - x_i)(x_j - x_{ij}) + \alpha\beta(\alpha-\beta)(x + x_i + x_j + x_{ij}) - \alpha\beta(\alpha-\beta)(\alpha^2 - \alpha\beta + \beta^2) = 0, \tag{Q_2}$$

$$\alpha(x - x_j)(x_i - x_{ij}) - \beta(x - x_i)(x_j - x_{ij}) + \delta\alpha\beta(\alpha-\beta) = 0. \tag{Q_1}$$

- All these equations are symmetric and possess the tetrahedron property.
- They appear as the nonlinear superposition principles for the Bäcklund transdormations of 1+1 PDE.

List of integrable equations

The degeneracy of biquadratics does not mean that the corresponding equation is trivial.

The properties of the following equations are the same as for the previous list:

$$\alpha(xx_i + x_jx_{ij}) - \beta(xx_j + x_ix_{ij}) + \delta(\alpha^2 - \beta^2) = 0, \qquad (H_3)$$

$$(x - x_{ij})(x_i - x_j) + (\beta - \alpha)(x + x_i + x_j + x_{ij}) + \beta^2 - \alpha^2 = 0, \qquad (H_2)$$

$$(x - x_{ij})(x_i - x_j) + \beta - \alpha = 0 \qquad (H_1)$$

However, there exist also a lot of asymmetric examples without tetrahedron property, and their classification is not completed.

Conclusion

- We have proven by inspection that, in nondegenerate case, any 3D-consistent triple can be brought to the shift-invariant and symmetric form and possesses the "tetrahedron property".
- The (Q_4) equation [5,6] contains 3 parameters which enumerate the orbits of general position of $(PSL_2(\mathbb{C}))^8$ action. It is therefore likely that any affine-linear equation can be considered as a member of some 3D-consistent triple.
- The classification in the degenerate case remains an open problem.

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Conclusion

- Several examples are known of 3D-consistent equations which are not affine-linear.
- Some noncommutative examples are known [7,8].
- Another version of 3D-consistency corresponds to Yang-Baxter maps with the unknowns on the edges of the lattice [9].
- Few 3D equations which are 4D-consistent are known [10].

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